GRAPH AND ROUTING DYNAMICS: MODELS AND ALGORITHMS
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HYPERBOLICITY AND GROMOV HYPERBOLIC GRAPHS

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1. Introduction.

We want to start this introduction with a cite from E. A. Jonckheere and P. Lohsoonthorn's paper [39] which illustrates the applications of hyperbolic graphs on the internet:

"One of the many security concerns in modern data networks is eavesdropping, that is, unauthorized packet interception along a link with the potential of reconstructing the full message − if all packets are sent along the same optimum path from source to destination, as TCP does under normal conditions. One of the proposed patches to such a security breach is to send packets in a randomnized fashion along different, nonoptimal routes [38]. Since the many routes have different delays, out-of-order packet arrival at the destination could create drops if the arrival sequence is altered by more than 3 slots. Unless some robustified TCP protocol is implemented, there is a need to restrict the paths to have costs bounded away from the optimum cost. On a graph, or on a surface or manifold for that matter, these near optimum paths may or may not remain within an identifiable neighborhood of the optimum path. In fact, in classical Riemannian geometry, this behavior is encapsulated in the concept of curvature: In a negative curvature space, the near optimal paths − referred to as quasi-geodesics − remain in a neighborhood of the optimal path, while in positive curvature space the near optimal paths could potentially spread across the whole manifold. For these facts to be applicable to graphs, there is a need to define a curvature concept for non-differentiable structures, a curvature concept upon which the features of Riemannian geometry can be extended to other structures. In what has been referred to as the most significant development in geometry over the past 20 years, the concept of negative curvature has been redefined in terms of such a more primitive concept as distance and has hence become applicable to graphs. Here we restrict ourselves to hyperbolic (negative curvature) graphs, the chief reason being that Monte Carlo simulation has indicated that the popular "growth, preferential attachment" model of Internet build up promotes negative curvature (see [39, Section 3.6]). Furthermore, from the point of view of network architecture, it appears desirable to design it hyperbolic, for the near optimal paths do not have to be sought across the whole network, but can be narrowed down to an identifiable neighborhood of the optimal path. The problem is that, while existence of bounds has been proved [33], they tend to be overly conservative, so that for engineering applications, tighter bounds must be sought."

In this course, we will try to understand the concepts related to hyperbolic graphs and solve this problem stated by E. A. Jonckheere and P. Lohsoonthorn by obtaining better bounds than the classical ones proved in [32] and [33].

2. A historical introduction to non-Euclidean geometries.

A part of this Section and of Section 4 appear in [20]; see this paper for more detailed information.

In this course, we are interested in the geometry of metric spaces which are negatively curved in some sense.

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Let us first briefly discuss the history of non-Euclidean geometries. Euclid's Elements consists of 13 books, written at about 300BC, that are mainly concerned with geometry (although they also contain some number theory and the method of exhaustion which is related to integration). It is the earliest known systematic discussion of geometry.

Book 1 begins with 23 definitions (of a point, line, etc.) and 10 axioms. Of these axioms, the following five are termed Postulates:

(1) Any two points can be joined by a straight line.

(2) Any straight line segment can be extended indefinitely in a straight line.

(3) Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

(4) All right angles are congruent.

(5) Parallel Postulate: If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

The Parallel Postulate is equivalent to the statement that for any given line R and point $p \notin R$, there is exactly one line through p that does not intersect R, i.e., that is parallel to R.

For two millenia, mathematicians were troubled by the Parallel Postulate of Euclid, principally because it is more complex and rather different from the other Postulates. For most of that time, mathematicians attempted to prove that it followed from the other postulates, as Proclus, Ibn al-Haytham (Alhacen), Omar Khayyám, Nasir al-Din al-Tusi, Witelo, Gersonides, Alfonso, and later Giovanni Gerolamo Saccheri, John Wallis, Johann Heinrich Lambert and Legendre. Some of them succeeded in finding a large variety of false "proofs" which all fail because they make some assumption that is equivalent to the Parallel Postulate. In fact, if we replace the Parallel Postulate by:

(a) for any line R and any point $p \notin R$, there exist at least two lines parallel to R passing through p, or

(b) for any line R and any point $p \notin R$, there exists no line parallel to R passing through p,

we obtain different geometries: hyperbolic geometry or elliptic geometry, respectively.

In elliptic geometry, whose main model is any sphere in $\mathbb{R}^3$, there are no parallel lines at all. Elliptic geometry has a variety of properties that differ from those of classical Euclidean plane geometry. For example, the sum of the angles of any triangle is always greater than π.

In the nineteenth century, hyperbolic geometry was extensively explored by Janos Bolyai and Nikolai Ivanovich Lobachevsky, after whom it sometimes is named. Lobachevsky published in 1830, while Bolyai independently discovered it and published in 1832. Carl Friedrich Gauss also studied hyperbolic geometry, describing in a 1824 letter to Taurinus that he had constructed it, but did not publish his work. Initially, some mathematicians thought that this new geometry was not consistent; however, Eugenio Beltrami provided models of the hyperbolic geometry in 1868, and used this to prove that hyperbolic geometry is consistent provided that Euclidean geometry is. The term "hyperbolic geometry" was introduced by Felix Klein in 1871. For more history, see [20], [29], [52] and [71].

There are four models commonly used for hyperbolic geometry: the Klein model, the Poincaré disc, the Poincaré halfplane, and the Lorentz model. These models define a real hyperbolic space which satisfies the axioms of a hyperbolic geometry. Despite their names, the first three mentioned above were introduced as models of hyperbolic space by Beltrami, not by Poincaré or Klein. We are mainly interested in the two Poincaré models.

The Poincaré metric in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is given infinitesimally at a point $z = x + iy \in \mathbb{D}$ by

$$ds_{\mathbb{D}}^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2} = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}\,, \qquad ds_{\mathbb{D}} = \frac{2\,|dz|}{1 - |z|^2}\,,$$

and so the hyperbolic area element is

$$dA_{\mathbb{D}} = \frac{4\,dx\,dy}{(1 - (x^2 + y^2))^2} = \frac{4\,dx\,dy}{(1 - |z|^2)^2}\,.$$

Given $z_1, z_2 \in \mathbb{D}$, the associated distance function is

$$d_{\mathbb{D}}(z_1, z_2) = 2 \operatorname{arctanh} \left| \frac{z_1 - z_2}{1 - z_1 \overline{z_2}} \right|.$$

The hyperbolic plane contains a unique geodesic between every pair of points. In the Poincaré disk $\mathbb{D}$, the geodesic lines are precisely the intersections with $\mathbb{D}$ of circles that cut the unit circle orthogonally, plus diameters of the boundary circle.

The Poincaré metric in the upper halfplane $\mathbb{U} = \{z = x + iy \in \mathbb{C} : y > 0\}$ is given infinitesimally at a point $z = x + iy \in \mathbb{U}$ by

$$ds_{\mathbb{U}}^2 = \frac{dx^2 + dy^2}{y^2}, \qquad ds_{\mathbb{U}} = \frac{|dz|}{y},$$

and so the hyperbolic area element is

$$dA_{\mathbb{U}} = \frac{4\, dx\, dy}{y^2}.$$

Given $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{U}$, the associated distance function satisfies

$$d_{\mathbb{U}}(z_1, z_2) = \log \frac{|z_1 - \overline{z_2}| + |z_1 - z_2|}{|z_1 - \overline{z_2}| - |z_1 - z_2|}, \quad \sinh^2 \frac{d_{\mathbb{U}}(z_1, z_2)}{2} = \frac{\cosh d_{\mathbb{U}}(z_1, z_2) - 1}{2} = \frac{|z_1 - z_2|^2}{4\, y_1\, y_2}.$$

The geodesic lines are precisely the intersections with $\mathbb{U}$ of circles orthogonal to the real line, plus rays perpendicular to the real line.

Both Poincaré models preserve hyperbolic angles, and are thereby conformal. All isometries within these models are therefore Möbius transformations. The halfplane model is "identical" (isometric) to the Poincaré disc model.

The area of a triangle in the hyperbolic plane increases more slowly and the area of a disk increases quicker than in the Euclidean setting. Let us now say more about both of these.

There is a simple and remarkable relationship between angles and area of a triangle which can be obtained as a consequence of Gauss-Bonnet formula:

The hyperbolic area of a triangle with interior angles α, β, γ is $\pi - (\alpha + \beta + \gamma)$. This holds even if one or more vertices of the triangle are on the ideal boundary (in which case the associated angles are zero). It follows from the Gauss-Bonnet formula that if we rescale upwards the sidelengths of a hyperbolic triangle, its area increases, with a limiting area of π as the sidelengths tend to infinity.

Then, Euclidean triangles are "wider" than hyperbolic triangles, and one can think that the Euclidean plane is "wider" than the hyperbolic plane.

The area A_r of a hyperbolic disk of radius r is independent of the center, and is given by $4\pi \sinh^2(r/2)$. The length L_r of the hyperbolic circle of radius r is $2\pi \sinh r$. Therefore, A_r and L_r are very similar to the corresponding Euclidean quantities when r is small:

$$A_r \approx \pi r^2, \qquad L_r \approx 2\pi r, \qquad \text{as } r \to 0^+.$$

However they increase far faster than in the Euclidean setting when r is large:

$$A_r \approx L_r \approx \pi e^r, \qquad \text{as } r \to \infty.$$

Hence, the hyperbolic plane is "wider" than the Euclidean plane (although Euclidean triangles are "wider" than hyperbolic ones).

There are many excellent books about hyperbolic geometry, e.g., the books by Anderson [6], Beardon [11] and Krantz [48].

In complex analysis, the most important property of the Poincaré metric is that holomorphic mappings are contractions with respect to it. More precisely, we have (see [2, p.3]):

Theorem 2.1 (Schwarz-Pick Lemma). *Every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ verifies*

$$d_{\mathbb{D}}\big(f(z_1), f(z_2)\big) \leqslant d_{\mathbb{D}}(z_1, z_2)$$

for every $z_1, z_2 \in \mathbb{D}$.

Furthermore, if the equality holds for some $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$, then f is an automorphism (i.e., a Möbius self-map of $\mathbb{D}$), and so $d_{\mathbb{D}}\big(f(z_1), f(z_2)\big) = d_{\mathbb{D}}(z_1, z_2)$ for every $z_1, z_2 \in \mathbb{D}$.

In fact, the Poincaré metric can be defined for any domain $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ has more than one point. If we denote by d_Ω the Poincaré distance in Ω, then we have the following generalization of Schwarz-Pick Lemma (see, e.g., the books [48] and [64]):

Theorem 2.2. *If $\partial\Omega_1$ and $\partial\Omega_2$ have more than one point and $f : \Omega_1 \to \Omega_2$ is holomorphic, then*

$$d_{\Omega_2}\big(f(z_1), f(z_2)\big) \leqslant d_{\Omega_1}(z_1, z_2)$$

for every $z_1, z_2 \in \Omega_1$.

Furthermore, if the equality holds for some $z_1, z_2 \in \Omega_1$ with $z_1 \neq z_2$, then f is a conformal map of Ω_1 onto Ω_2, and so $d_{\Omega_2}\big(f(z_1), f(z_2)\big) = d_{\Omega_1}(z_1, z_2)$ for every $z_1, z_2 \in \Omega_1$.

The simpler particular case of Schwarz-Pick Lemma is the classical Schwarz's Lemma (see [2, p.3]):

Theorem 2.3. *Every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ verifies*

$$\big|f(z)\big| \leqslant |z|$$

for every $z \in \mathbb{D}$.

This is a bound for the growth of *every* holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ (with the normalization $f(0) = 0$).

One of the most famous applications of Theorem 2.2 is the following (see [1] or [2, p.19]):

Theorem 2.4 (Schottky's Theorem). *If $f : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0, 1\}$ is holomorphic, then*

$$|f(z)| \leqslant \exp\Big(\frac{1 + |z|}{1 - |z|} \big(7 + \max\big\{0, \log |f(0)|\big\}\big) \Big) ,$$

for every $z \in \mathbb{D}$.

Note that this is a bound for the growth of *every* holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0, 1\}$.

3. Definition of Gromov hyperbolic spaces.

Now, let us introduce the concept of Gromov hyperbolicity and the main results concerning this theory. For detailed expositions about Gromov hyperbolicity, see e.g. [3], [32], [28], [18, II.H] or [70].

Gromov hyperbolicity was introduced by Gromov in the setting of geometric group theory [33], [34], [32], [28], but has played an increasing role in analysis on general metric spaces [16], [17], [8], with applications to the Martin boundary, invariant metrics in several complex variables [7] and extendability of Lipschitz mappings [49].

Hyperbolic spaces play an important role in geometric group theory and in geometry of negatively curved spaces (see [3], [32], [33]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [3], [32], [33]).

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [33] and the references therein), where it was demonstrated to have a practical importance. This theory was applied principally to the study of automatic groups (see [55]), which play a role in the science of computation (indeed, hyperbolic groups are strongly geodesically automatic, that is, there is an automatic structure on the group, where the language accepted by the word acceptor is the set of all geodesic words [26]).

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. As we have seen in the Introduction, another important application of these spaces is secure transmission of information on the internet (see [39], [40], [41]). Furthermore, the hyperbolicity plays an important role in the spread of viruses through the network (see [40], [41]). Ideas related to hyperbolicity have been applied in numerous other networks applications, e.g., to problems such as distance estimation, sensor networks, and traffic flow and congestion minimization [68], [44], [45], [54], [10], as well as large-scale data visualization [53]. The latter applications typically take important advantage of the idea that data are often hierarchical or tree-like and that there is Şmore roomŤ in hyperbolic spaces of a given dimension than corresponding Euclidean spaces. The hyperbolicity is also useful in the study of DNA data (see [19]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory (see, e.g., [10], [12], [13], [14], [19], [22], [23], [24], [27], [31], [39], [40], [41], [42], [43], [44], [45], [47], [50], [51], [53], [54], [56], [57], [58], [59], [62], [63], [65], [66], [67], [68], [69], [73]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [35]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [8], [16], [36], [37], [59], [60], [61], [66], [67]. In particular, in [59], [66], [67], [69] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a very simple graph; hence, it is useful to study hyperbolic graphs from this point of view.

Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. We define the *length* of γ as

$$L(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\} .$$

We say that γ is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. A *geodesic line* is a geodesic whose domain is $\mathbb{R}$.

We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If the metric space X is a graph, we use the notation $[u, v]$ for the edge joining the vertices u and v.

By a graph G we mean a set of points called vertices connected by (undirected) edges; the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$; we assume also that each edge has a length assigned to it. In order to consider a graph G as a geodesic metric space, identify (by an isometry) any edge $[u, v] \in E(G)$ with the real interval $[0, l]$ (if $l := L([u, v])$); therefore, any point in the interior of an edge is a point of G. Then G is naturally equipped with a distance defined on its points, induced by taking the shortest paths in G, and we see G as a metric graph.

We allow loops and multiple edges in the graphs; we also allow edges of arbitrary lengths. We always consider graphs which are connected and locally finite (i.e., in each ball there are just a finite number of edges). These properties guarantee that the graphs are geodesic metric spaces. In what follows, although the results will be stated for hyperbolic spaces, you can think that they are always hyperbolic graphs, by simplicity.

Recall that if X is a metric space, $x \in X$ and $E \subseteq X$, the distance $d(x, E)$ is defined as $d(x, E) := \inf \big\{ d(x, y) | \, y \in E \big\}$.

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon with sides $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leqslant \delta$. In other words, a polygon is δ-thin if each of its sides is contained in the δ-neighborhood of the union of the other sides. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geqslant 0 | \, J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by

$\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) \,|\, T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geqslant 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geqslant 0 \,|\, X \text{ is } \delta\text{-hyperbolic}\}$.

One can check that if X is a δ-hyperbolic geodesic metric space, then every geodesic polygon with $n \geqslant 3$ sides is $(n-2)\delta$-thin.

Examples:

1. Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic.

2. The real line $\mathbb{R}$ is 0-hyperbolic: In fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore any geodesic triangle in $\mathbb{R}$ is 0-thin.

3. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic, since the midpoint of a side on a large equilateral triangle is far from all points on the other two sides.

These arguments can be applied to higher dimensions:

4. A normed real vector space is hyperbolic if and only if it has dimension 1.

5. Every metric tree with arbitrary length edges is 0-hyperbolic, by the same reason that the real line.

6. The unit disk $\mathbb{D}$ (with its Poincaré metric) is $\log(1+\sqrt{2}\,)$-thin: Consider any geodesic triangle T in $\mathbb{D}$. It is clear that T is contained in an ideal triangle T', all of whose sides are of infinite length, with $\delta(T) \leqslant \delta(T')$. Since all ideal triangles are isometric, we can consider just one fixed T'. Then, a computation gives $\delta(T') = \log(1+\sqrt{2}\,)$.

7. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -c^2 < 0$, for some constant c, is hyperbolic (see, e.g., [32, p.52]).

8. The graph Γ of the routing infrastructure of the Internet is also empirically shown to be hyperbolic (see [9]). One can think that this is a trivial (and then a non-useful) fact, since every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. The point is that the quotient

$$\frac{\delta(\Gamma)}{\operatorname{diam}\Gamma}$$

is very small, and this makes the tools of hyperbolic spaces applicable to Γ (see, e.g., [24]).

As a remark, the main examples of hyperbolic graphs are trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [25]).

It is worth pointing out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Note that, first of all, one needs to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then, to take the supremum over all the possible choices for P and then over all the possible choices for T. Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that the location of geodesics in the space is not usually known.

If X is a metric space, we define the *Gromov product* of $x, y \in X$ with base point $w \in X$ by

$$(x,y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big).$$

A geometric interpretation of the Gromov product is obtained by mapping the triple (x,y,w) isometrically onto a triple (x',y',w') in the Euclidean plane. The circle inscribed to the triangle x',y',w' meets the sides $[w'x']$ and $[w'y']$ at points x^* and y^*, respectively, and we have $(x,y)_w = |x^* - w'| = |y^* - w'|$.

If X is a Gromov hyperbolic space, it holds

$$(x,z)_w \geqslant \min\big\{(x,y)_w, (y,z)_w\big\} - \delta \tag{3.1}$$

for every $x, y, z, w \in X$ and some constant $\delta \geqslant 0$ (see, e.g., [3, Proposition 2.1], [32, p.41] or [70, 2.34,2.35]). Let us denote by $\delta^*(X)$ the sharp constant for this inequality, i.e.,

$$\delta^*(X) := \sup \big\{ \min \big\{(x,y)_w, (y,z)_w\big\} - (x,z)_w \,:\, x,y,z,w \in X \big\}.$$

Remark 3.2. If X is a geodesic metric space, it is known that (3.1) is, in fact, equivalent to our definition of Gromov hyperbolicity; furthermore, $\delta^*(X) \leqslant 3\,\delta(X)$ and $\delta(X) \leqslant 3\,\delta^*(X)$ (see, e.g., [70, 2.34,2.35]).

Then (3.1) extends the definition of Gromov hyperbolicity to the context of (non-necessarily geodesic) metric spaces. The disadvantage of the Gromov product definition is that its geometric meaning is unclear at first sight, whereas the thin triangles definition is very easy to understand geometrically.

The following useful estimate is the key to understand the geometric meaning of the Gromov product definition (3.1):

Proposition 3.3. ([32, Lemma 1.7, p.38]) *If X is a δ-hyperbolic geodesic metric space, then for every $x, y, w \in X$ and every geodesic $[xy]$, we have*

$$d(w,[xy]) - 4\delta \leqslant (x,y)_w \leqslant d(w,[xy]).$$

Indeed only the lower bound requires hyperbolicity.

We would like to recall the surprising geometric interpretation of the Gromov product in the hyperbolic plane. Like in the Euclidean setting, there exists a relation among the three sides of a right-angled triangle in the hyperbolic plane (the hyperbolic Pythagorean theorem):

$$\cosh c = \cosh a \, \cosh b.$$

We also have the hyperbolic Cosine rule for any hyperbolic triangle:

$$\cosh c = \cosh a \, \cosh b - \sinh a \, \sinh b \, \cos\theta.$$

If a, b, c are large, then this latter formula is asymptotically equivalent to

$$\frac{1}{2} e^c \approx \frac{1}{4} e^{a+b} (1 - \cos\theta),$$

and we deduce

$$e^c \approx e^{a+b} \sin^2(\theta/2) \qquad \Longrightarrow \qquad \frac{1}{2}(a+b-c) \approx \log \frac{1}{\sin(\theta/2)}\,.$$

If x, y, w are the vertices of this hyperbolic triangle and θ is the angle at w, then we have

$$(x,y)_w \approx \log \frac{1}{\sin(\theta/2)}\,,$$

and we can determine (approximately) the angle θ in terms of the Gromov product.

Then we could expect that the Gromov product allows to estimate "something like angles" in hyperbolic spaces (for instance, in hyperbolic graphs!); this is the case and, consequently, the hyperbolic spaces have richer structure than the general metric spaces.

In the setting of linear spaces with inner product, the main angle is $\pi/2$; however, in hyperbolic spaces the main angle is π: One can check that, in a geodesic metric space, $(x,y)_w = 0$ if and only if w belongs to a geodesic joining x and y.

The following result is a good example of "angle estimation" in hyperbolic spaces:

Theorem 3.4. ([32, Theorem 16, p.92]) *Let us consider constants $\delta \geqslant 0, r, \ell > 0$, a δ-hyperbolic geodesic metric space X and a finite sequence $\{x_j\}_{0 \leqslant j \leqslant n}$ in X with*

$$\begin{aligned} &d(x_{j-1},x_{j+1}) \geqslant \max\{d(x_{j-1},x_j), d(x_j,x_{j+1})\} + 18\delta + r, \qquad \text{for every } 0 < j < n,\\ &d(x_{j-1},x_j) \leqslant \ell, \qquad \text{for every } 0 < j \leqslant n. \end{aligned}$$

Then $[x_0x_1] \cup [x_1x_2] \cup \cdots \cup [x_{n-1}x_n]$ is an $(\alpha,0)$-quasigeodesic, with $\alpha := \max\{\ell,\, 1/r\}$.

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \ge 1,\ \beta \ge 0$ if, for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \le d_Y(f(x), f(y)) \le \alpha d_X(x, y) + \beta.$$

The function f is ε-*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \le \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \ge 1,\ \beta, \varepsilon \ge 0$ such that f is an ε-full (α, β)-quasi-isometric embedding.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \longrightarrow Y$. It is quite easy to see that being quasi-isometric is an equivalence relation.

An (α, β)-*quasigeodesic* of a metric space X is an (α, β)-quasi-isometric embedding $\gamma : I \longrightarrow X$, where I is an interval of $\mathbb{R}$. Note that a $(1, 0)$-quasigeodesic is a geodesic.

Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call ε-*neighborhood* of Y in X, denoted by $V_\varepsilon(Y)$ to the set $\{x \in X : d_X(x, Y) \le \varepsilon\}$. The *Hausdorff distance* between two subsets Y and Z of X, denoted by $\mathcal{H}(Y, Z)$, is the number defined by:

$$\inf\{\varepsilon > 0 : Y \subset V_\varepsilon(Z) \textit{ and } Z \subset V_\varepsilon(Y)\}.$$

4. Gromov hyperbolicity, Mathematical Analysis and Geometry.

The ideal boundary of a metric space is a type of boundary at infinity which is a very useful concept when dealing with negatively curved spaces. We want to talk about some subjects in which this boundary is useful.

A main problem in the study of Partial Differential Equations in Riemannian Manifolds is whether or not there exist nonconstant bounded harmonic functions. A way to approach this problem is to study whether the so-called Dirichlet problem at infinity (or the asymptotic Dirichlet problem) is solvable on a complete Riemannian manifold M. That is to say, raising the question as to whether every continuous function on the boundary ∂M has a (unique) harmonic extension to M. Of course, the answer, in general, is no, since the simplest manifold $\mathbb{R}^n$ admits no positive harmonic functions other than constants. However, the answer is positive for the unit disk $\mathbb{D}$.

In [4] Ancona studied the asymptotic Dirichlet problem on Gromov hyperbolic graphs and in [5] on Gromov hyperbolic Riemannian manifolds with bounded geometry and a positive lower bound $\lambda_1(M) > 0$ for Dirichlet eigenvalues. In the papers [21] and [46] conditions on Gromov hyperbolic manifolds M that imply the positivity of $\lambda_1(M)$ are given and, consequently, the Dirichlet problem is solvable for many Gromov hyperbolic manifolds.

One of the most important features of the transition from a Gromov hyperbolic space to its Gromov boundary is that it is functorial. If $f : X \longrightarrow Y$ is in a certain class of maps between two Gromov hyperbolic spaces X and Y, then there is a boundary map $\partial f : \partial X \longrightarrow \partial Y$ which is in some other class of maps. In particular, if f is a quasi-isometry, then ∂f is a bihölder map (with respect to the Gromov metric on the boundary).

It is well known that biholomorphic maps between domains (with smooth boundaries) in $\mathbb{C}$ can be extended as a homeomorphism between their boundaries. If we consider domains in $\mathbb{C}^n$ $(n > 1)$ instead in $\mathbb{C}$, then the problem is very difficult. C. Fefferman (Fields medallist) showed in Inventiones Mathematicae (see [30]), with a very long and technical proof, that biholomorphic maps between bounded strictly pseudoconvex domains with smooth boundaries can be extended as a homeomorphism between their boundaries. It is possible to give a “more elementary” proof of this extension result using the functoriality of Gromov hyperbolic spaces: If we consider the Carathéodory metric on a bounded smooth strictly pseudoconvex domain in $\mathbb{C}^n$, then it is Gromov hyperbolic, and the Gromov boundary is homeomorphic to the topological boundary (see [7]). Since any biholomorphic map f between such two domains is an isometry for the Carathéodory metrics, the boundary map ∂f is essentially a boundary extension of f that is a homeomorphism between the boundaries (in fact, it is bihölder with respect to the Carnot-Carathéodory metrics in the boundaries). Fefferman’s result gives much more precise information, but this last proof is simpler and gives information about a class of maps that is much more general than biholomorphic maps: the quasi-isometries for the Carathéodory metrics.

In applications to various areas of mathematics, the Gromov boundary can be similarly be proved (under appropriate conditions) to coincide with other "finite" boundaries, such as the Euclidean or inner Euclidean boundary, or the Martin boundary, so we obtain a variety of boundary extension results as above.

Isometries (and quasi-isometries) in a hyperbolic space X can be extended (as an homeomorphism) to the Gromov boundary ∂X of the space. This fact allows to classify the isometries as *hyperbolic*, *parabolic* and *elliptic*, like the Möbius maps in $\mathbb{D}$, in terms of their fixed points in $X \cup \partial X$.

There are just three possibilities:

- There are exactly two fixed points in $X \cup \partial X$ and both are in ∂X (hyperbolic isometry).
- There is a single fixed point in $X \cup \partial X$ and it is in ∂X (parabolic isometry).
- There is a single fixed point in $X \cup \partial X$ and it is in X (elliptic isometry).

A main ingredient in the proof of this result in the unit disk $\mathbb{D}$ is that the isometries are holomorphic functions. Surprisingly, the tools in hyperbolic spaces provide a new and general proof just in terms of distances!

5. Main results.

We state now some of the main facts about hyperbolic spaces.

In the study of any mathematical property, the class of maps which preserve that property plays a central role in the theory. The following result shows that quasi-isometries preserve hyperbolicity.

Theorem 5.1 (Invariance of hyperbolicity)**.** *Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic.*

Besides, if f is ε-full for some $\varepsilon \geqslant 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

We next discuss the connection between hyperbolicity and *geodesic stability.* In the complex plane (with its Euclidean distance), there is only one optimal way of joining two points: a straight line segment. However if we allow "limited suboptimality", the set of "reasonably efficient paths" (quasi-geodesics) are well spread. For instance, if we split the circle $\partial D(0, R) \subset \mathbb{C}$ into its two semicircles between the points R and $-R$, then we have two reasonably efficient paths (two $(\pi/2, 0)$-quasigeodesics) between these endpoints such that the point Ri on one of the semicircles is far from all points on the other semicircle provided that R is large. Even an additive suboptimality can lead to paths that fail to stay close together. For instance, the union of the two line segments in $\mathbb{C}$ given by $[0, R+i\sqrt{R}\,]$ and $[R+i\sqrt{R}\,, 2R]$ gives a path of length less than $2R+1$ (since $2\sqrt{R^2+R} \leqslant 2R+1$), and so is "additively inefficient" by less than 1 (a $(1, 1)$-quasigeodesic). However, its corner point is very far from all points on the line segment $[0, 2R]$ when R is very large.

The situation in Gromov hyperbolic spaces is very different, since all such reasonably efficient paths ((α, β)-quasigeodesics for fixed α, β) stay within a bounded distance of each other:

Theorem 5.2 (Geodesic stability)**.** *For any constants $\alpha \geqslant 1$ and $\beta, \delta \geqslant 0$ there exists a constant $H = H(\delta, \alpha, \beta)$ such that for every δ-hyperbolic geodesic metric space and for every pair of (α, β)-quasigeodesics g, h with the same endpoints, $\mathcal{H}(g, h) \leq H$.*

The geodesic stability is not just a useful property of hyperbolic spaces; in fact, M. Bonk proves in [15] that the geodesic stability is equivalent to the hyperbolicity:

Theorem 5.3. ([15, p.286]) *Let X be a geodesic metric space with the following property: For each $a \geqslant 1$ there exists a constant H such that for every $x, y \in X$ and any $(a, 0)$-quasigeodesic g in X starting in x and finishing in y there exists a geodesic γ joining x and y satisfy $\mathcal{H}(g, \gamma) \leqslant H$. Then X is hyperbolic.*

Theorem 5.2 allows to prove Theorem 5.1:

Proof of Theorem 5.1. By hypothesis there exists $\delta \geqslant 0$ such that Y is δ-hyperbolic.

Let T be a geodesic triangle in X with sides g_1, g_2 y g_3, and T_Y the triangle with (α, β)-quasigeodesic sides $f(g_1)$, $f(g_2)$ y $f(g_3)$ in Y. Let γ_j be a geodesic joining the endpoints of $f(g_j)$, for $j = 1, 2, 3$, and T' the geodesic triangle in Y with sides $\gamma_1, \gamma_2, \gamma_3$.

Let p be any point in $f(g_1)$. We are going to prove that there exists a point $q \in f(g_2) \cup f(g_3)$ with $d_Y(p, q) \leqslant K$, where $K := \delta + 2H(\delta, \alpha, \beta)$. By Theorem 5.2, there exists a point $p' \in \gamma_1$ with $d_Y(p, p') \leqslant H(\delta, \alpha, \beta)$. Since T' is a geodesic triangle, it is δ-thin and there exists $q' \in \gamma_2 \cup \gamma_3$ with $d_Y(p', q') \leqslant \delta$. Using again Theorem 5.2, there exists a point $q \in f(g_2) \cup f(g_3)$ con $d_Y(q, q') \leqslant H(\delta, \alpha, \beta)$. Therefore,

$$\begin{aligned} d_Y(p, f(g_2) \cup f(g_3)) &\leqslant d_Y(p, q) \leqslant d_Y(p, p') + d_Y(p', q') + d_Y(q', q) \\ &\leqslant H(\delta, \alpha, \beta) + \delta + H(\delta, \alpha, \beta). \end{aligned}$$

Let $z \in T$; without loss of generality we can assume that $z \in g_1$. We have seen that there exists a point $q \in f(g_2) \cup f(g_3)$ with $d_Y(f(z), q) \leqslant K$. If $w \in g_2 \cup g_3$ satisfies $f(w) = q$, then

$$d_X(z, g_2 \cup g_3) \leqslant d_X(z, w) \leqslant \alpha d_Y(f(z), q) + \alpha\beta \leqslant \alpha K + \alpha\beta.$$

Hence, T is $(\alpha K + \alpha\beta)$-thin. Since T is an arbitrary geodesic triangle, X is $(\alpha\delta + 2\alpha H(\delta, \alpha, \beta) + \alpha\beta)$-hyperbolic.

Assume now that f is ε-full. One can check that an "inverse" quasi-isometry $f^- : Y \longrightarrow X$ can be constructed as follows: for $y \in Y$ choose $x \in X$ with $d_Y(f(x), y) \leqslant \varepsilon$ and define $f^-(y) := x$. Then the first part of the Theorem gives the result. □

We will need the following result.

Lemma 5.4. *Let X be a δ-hyperbolic geodesic metric space and $x, y, z \in X$ with $d(x, [yz]) = d(x, y)$. Then,*

$$d(x, y) + d(y, z) - 8\delta \leqslant d(x, z) \leqslant d(x, y) + d(y, z).$$

Proof. The upper bound for $d(x, z)$ is just the triangle inequality; let us prove the lower bound.

By Proposition 3.3, we have

$$d(x, y) = d(x, [yz]) \leqslant 4\delta + (y, z)_x = 4\delta + \frac{1}{2}\big(d(y, x) + d(z, x) - d(y, z)\big),$$

and we deduce the result. □

The following result is a weak version of the geodesic stability theorem (Theorem 5.2). This version has an advantage: it provides very simple and accurate constants. Then it provides a solution to the problem stated by E. A. Jonckheere and P. Lohsoonthorn in [39].

Theorem 5.5. ([62, Theorem 2.9]) *Let X be a δ-hyperbolic geodesic metric space, $u, v \in X$, b a non-negative constant, h a curve joining u and v with $L(h) \leqslant d(u, v) + b$, and g any geodesic joining u and v. Then,*

$$h \subseteq V_{8\delta+b/2}(g), \qquad g \subseteq V_{16\delta+b}(h), \qquad \mathcal{H}(g, h) \leqslant 16\delta + b.$$

Proof. Fix $x \in h$ and let $y \in g$ a point with $d(x, y) = d(x, g)$. By hypothesis, $L(h) \leqslant L(g) + b = d(u, v) + b$. Since h joins u and v, and $x \in h$, we have $L(h) \geqslant d(u, x) + d(x, v)$.

Lemma 5.4 allows to deduce

$$\begin{aligned} d(u, v) &\geqslant L(h) - b \geqslant d(u, x) + d(x, v) - b \\ &\geqslant d(u, y) + d(y, x) - 8\delta + d(x, y) + d(y, v) - 8\delta - b \\ &= d(u, v) + 2d(x, y) - 16\delta - b. \end{aligned}$$

Therefore, $d(x, g) = d(x, y) \leqslant 8\delta + b/2$ and, since x is an arbitrary point in h, we deduce $h \subseteq V_{8\delta+b/2}(g)$.

Fix now $\varepsilon > 0$ and choose points $x_0 = u, x_1, \dots, x_k = v$ in h with $d(x_j, x_{j+1}) \leqslant \varepsilon$ for $j = 0, 1, \dots, k-1$. Denote by y_j a nearest point from x_j in g for each $j = 0, 1, \dots, k$. The previous argument, with $x = x_j$ gives $d(x_j, y_j) \leqslant 8\delta + b/2$ for $j = 0, 1, \dots, k$. Hence,

$$\begin{aligned} d(y_j, y_{j+1}) &\leqslant d(y_j, x_j) + d(x_j, x_{j+1}) + d(x_{j+1}, y_{j+1}) \\ &\leqslant 8\delta + b/2 + \varepsilon + 8\delta + b/2 \\ &= 16\delta + b + \varepsilon. \end{aligned}$$

Since $y_0 = x_0 = u$ and $y_k = x_k = v$, for each point $z \in g$ there exists $y_j \in g$ with $d(z, y_j) \leqslant 8\delta + b/2 + \varepsilon/2$. Then,

$$d(z, h) \leqslant d(z, x_j) \leqslant d(z, y_j) + d(y_j, x_j) \leqslant 8\delta + b/2 + \varepsilon/2 + 8\delta + b/2 = 16\delta + b + \varepsilon/2.$$

Consequently, $d(z, h) \leqslant 16\delta + b$, and we deduce $g \subseteq V_{16\delta+b}(h)$.

Finally, $\mathcal{H}(g, h) \leqslant 16\delta + b$ follows from $h \subseteq V_{8\delta+b/2}(g)$ and $g \subseteq V_{16\delta+b}(h)$. □

6. Proof of Proposition 3.3.

As we have seen, there are several equivalent definitions of Gromov hyperbolicity. In order to proof Proposition 3.3, we need the definition of fine triangles.

Definition 6.1. Given a geodesic triangle $T = \{x, y, z\}$ in a geodesic metric space X, let T_E be a Euclidean triangle with sides of the same length than T. Since there is no possible confusion, we will use the same notation for the corresponding points in T and T_E. The maximum inscribed circle in T_E meets the side $[xy]$ (respectively, $[yz]$, $[zx]$) in a point z' (respectively, x', y') such that $d(x, z') = d(x, y')$, $d(y, x') = d(y, z')$ and $d(z, x') = d(z, y')$. We call the points x', y', z', the *internal points* of $\{x, y, z\}$. There is a unique isometry f_T of the triangle $\{x, y, z\}$ onto a tripod (a star graph with one vertex w of degree 3, and three vertices x_0, y_0, z_0 of degree one, such that $d(x_0, w) = d(x, z') = d(x, y')$, $d(y_0, w) = d(y, x') = d(y, z')$ and $d(z_0, w) = d(z, x') = d(z, y')$). The triangle $\{x, y, z\}$ is *δ-fine* if $f_T(p) = f_T(q)$ implies that $d(p, q) \leqslant \delta$. The space X is *δ-fine* if every geodesic triangle in X is δ-fine.

A basic result is that hyperbolicity is equivalent to be fine:

Theorem 6.2. ([32, Proposition 2.21, p.41]) *Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-fine.*
(2) *If X is δ-fine, then it is δ-hyperbolic.*

Proposition 6.3. *If X is a δ-fine geodesic metric space, then for every $x, y, z \in X$ and every geodesic $[xy]$, we have*

$$d(z, [xy]) - \delta \leqslant (x, y)_z \leqslant d(z, [xy]).$$

Indeed only the lower bound requires hyperbolicity.

Proof. Let x', y', z' be the internal points of the geodesic triangle $T := \{x, y, z\}$; then $(x, y)_z = d(z, x') = d(z, y')$.

Let $w \in [xy]$ with $d(z, w) = d(z, [xy])$, and $w^* \in [xz] \cup [yz]$ with $f_T(w) = f_T(w^*)$. Without loss of generality we can assume that $w^* \in [xz]$. Hence,

$$(x, y)_z = d(z, y') \leqslant d(z, w^*) = d(z, x) - d(x, w^*) = d(z, x) - d(x, w) \leqslant d(z, w) = d(z, [xy]).$$

Furthermore, since T is δ-fine,

$$d(z, [xy]) \leqslant d(z, z') \leqslant d(z, y') + d(y', z') \leqslant (x, y)_z + \delta.$$

□

Proposition 3.3 follows from Theorem 6.2 and Proposition 6.3.

7. Exercises.

1. Show that $d_p(x,y) := |x-y|^p$ is a distance in $\mathbb{R}$ for each $0 < p \leqslant 1$ and that it is not a distance for $p > 1$.

2. Show that $(\mathbb{R}, d_p)$, with $d_p(x,y) := |x-y|^p$, is a geodesic metric space if and only if $p = 1$.

3. It is well-known that $\mathbb{R}^2$ is a metric space with the distance

$$d\big((x_1,x_2),(y_1,y_2)\big) := \big(|x_1-y_1|^p + |x_2-y_2|^p\big)^{1/p}$$

for each $p \geqslant 1$. For what values of $p \geqslant 1$ is $(\mathbb{R}^2, d)$ a geodesic metric space?

4. Let X be any metric space. Show that for every $x,y,z,w \in X$ we have:
(1) $(x,y)_w = (y,x)_w$, $(x,y)_y = (x,y)_x = 0$.
(2) $d(x,y) = (x,z)_y + (y,z)_x$.
(3) $0 \leqslant (x,y)_w \leqslant \min\big\{d(x,w),\, d(y,w)\big\}$.
(4) $\big|(x,y)_w - (x,y)_z\big| \leqslant d(w,z)$.
(5) $\big|(x,y)_w - (x,z)_w\big| \leqslant d(y,z)$.
(6) If $x \in [yw]$, then $(x,y)_w = d(x,w)$.

5. Show that if (X,d) is a geodesic metric space, then $(x,y)_w = 0$ if and only if w belongs to a geodesic joining x and y.

6. Show that if X is a δ-hyperbolic geodesic metric space, then every geodesic polygon P with n sides ($n \geqslant 3$) is $(n-2)\delta$-thin, i.e., if p belongs to a side of P then the distance from p to the union of the other $n-1$ sides of P is less than or equal to $(n-2)\delta$.

7. Let $f: X \longrightarrow Y$ be an (a,b)-quasi-isometric embedding between the metric spaces X and Y, and γ a geodesic in X. Show that $f(\gamma)$ is an (a,b)-quasigeodesic in Y.

8. Show that if $f: X \to Y$ is an (a,b)-quasi-isometric embedding and $g: Y \to Z$ is a (c,d)-quasi-isometric embedding, then the composition $g \circ f: X \to Z$ is also a quasi-isometric embedding.

9. If $f: X \to Y$ is an ε-full (a,b)-quasi-isometry between two metric spaces, find an (a',b')-quasi-isometry $g: Y \to X$, where a', b' depend just on a, b and ε (we say that g is a *quasi-inverse* of f).

10. Show that the unit disk $\mathbb{D}$ and the upper halfplane $\mathbb{U}$ (with their respective Poincaré metrics) are $\log(1+\sqrt{2}\,)$-thin. *Hint: Show that it suffices to consider the ideal triangle in* $\mathbb{U}$ *with vertices* $0, 1, \infty \in \partial\mathbb{U}$ *and that, if* I *is the imaginary axis and* $z = re^{i\theta} \in \mathbb{U}$*, then* $\tanh d_{\mathbb{U}}(z, I) = \cos\theta$ *(see, e.g.,* [11, p.162]*).*

11. Let us consider a δ-hyperbolic geodesic metric space and a triangle $T \subseteq X$ with (a,b)-quasigeodesics sides. Show that T is K-thin, with $K = \delta + 2H(\delta,a,b)$, where H is the constant in Theorem 5.2.

12. Show that if h is any curve joining two points x,y in a geodesic metric space with $L(h) \leqslant d(x,y)+b$, then h is a $(1,b)$-quasigeodesic.

13. Find two metric spaces $(X,d_X),(Y,d_Y)$ and an (a,b)-quasi-isometric embedding $f: X \to Y$ such that Y is hyperbolic and X is not hyperbolic. Is that a contradiction?

14. Let us consider a $(1,b)$-quasi-isometric embedding between two metric spaces $f: X \longrightarrow Y$. Show that if Y is δ-hyperbolic with respect to the Gromov product definition, then X is $(\delta+3b)$-hyperbolic with respect to the Gromov product definition.

15. Let us consider an ε-full $(1,b)$-quasi-isometry between two metric spaces $f: X \longrightarrow Y$. Show that if X is δ-hyperbolic, then Y is δ'-hyperbolic, and find an explicit expression for δ'.

16. Let X be a geodesic metric space. Show that if X is δ-fine, then it is δ-hyperbolic.

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