Hyperbolicity and Gromov hyperbolic graphs

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INTRODUCTION

We want to start with some ideas from E. A. Jonckheere and P. Lohsoonthorn's paper [A hyperbolic geometry approach to multipath routing, 2002] which illustrates the applications of hyperbolic graphs on the internet:

In order to preserve the security, when we send information on the internet, one of the classical proposed strategies is to split the message into packets and send these packets in a randomnized fashion along different, non-optimal paths.

In order to restore the message, it is useful to receive the packets in (approximately) the same order in which they were sent.

Then, there is a need to restrict the paths to have costs close to the optimum cost.

On a graph (or on a surface), these near optimum paths may or may not remain within an identifiable neighborhood of the optimum path.

In classical Riemannian geometry, this behavior is encapsulated in the concept of curvature: In a space with negative curvature, the near optimal paths (quasi-geodesics) remain in a neighborhood of the optimal path.

For these facts to be applicable to our problem, there is a need to define a curvature concept for non-differentiable structures, as graphs.

In what has been referred to as the most significant development in geometry over the past 20 years, the concept of negative curvature has been redefined in terms of such a more primitive concept as distance and has hence become applicable to graphs.

Monte Carlo simulation has indicated that the popular "growth, preferential attachment" model of Internet build up promotes negative curvature.

Furthermore, from the point of view of network architecture, it appears desirable to design it to be hyperbolic, for the near optimal paths do not have to be sought across the whole network, but can be narrowed down to an identifiable neighborhood of the optimal path.

The problem is that, whereas existence of bounds (for the radii of these neighborhoods) has been proved, they tend to be overly conservative, so that for engineering applications, tighter bounds must be sought.

In this course:

• we will try to understand the concepts related to hyperbolic graphs,

• we will solve this problem stated by E. A. Jonckheere and P. Lohsoonthorn by obtaining better bounds than the classical ones.

NON-EUCLIDEAN GEOMETRIES

Euclid's Elements consists of 13 books, written at about 300BC, that are mainly concerned with geometry. It is the earliest known systematic discussion of geometry.

Book 1 begins with 23 definitions and 10 axioms.

One of these axioms is the Parallel Postulate:

> For any given line R and point $p \notin R$, there is exactly one line through p that does not intersect R, i.e., that is parallel to R.

For most of two millenia, mathematicians attempted to prove that the Parallel Postulate followed from the other axioms.

Some of them succeeded in finding a large variety of false "proofs".

In fact, if we replace the Parallel Postulate by:

(a) for any line R and any point $p \notin R$, there exist at least two lines parallel to R passing through p,

or

(b) for any line R and any point $p \notin R$, there exists no line parallel to R passing through p,

we obtain different geometries: hyperbolic geometry or elliptic geometry, respectively.

In elliptic geometry, whose main model is any sphere in $\mathbb{R}^3$, there are no parallel lines at all.

Elliptic geometry has a variety of properties that differ from those of classical Euclidean plane geometry. For example, the sum of the angles of any triangle is always greater than π.

In the nineteenth century, hyperbolic geometry was extensively explored by Bolyai, Lobachevsky and Gauss.

Initially, some mathematicians thought that this new geometry was not consistent. However, Beltrami proved that hyperbolic geometry is consistent provided that Euclidean geometry is.

There are four models commonly used for hyperbolic geometry: the Klein model, the Poincaré disc, the Poincaré halfplane, and the Lorentz model. These models define a real hyperbolic space which satisfies the axioms of a hyperbolic geometry.

We are mainly interested in the two Poincaré models.

The Poincaré metric in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is given infinitesimally at a point $z = x + iy \in \mathbb{D}$ by

$$ds_{\mathbb{D}}^2 = \frac{4\,(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2}\,, \qquad ds_{\mathbb{D}} = \frac{2\,|dz|}{1 - |z|^2}\,.$$

Given $z_1, z_2 \in \mathbb{D}$, the associated distance function is

$$d_{\mathbb{D}}(z_1, z_2) = 2\,\text{arctanh}\left|\frac{z_1 - z_2}{1 - z_1\overline{z_2}}\right|.$$

The hyperbolic plane contains a unique geodesic between every pair of points.

In the Poincaré disk $\mathbb{D}$, the geodesic lines are precisely the intersections with $\mathbb{D}$ of circles that cut the unit circle orthogonally, plus diameters of the boundary circle.

The Poincaré metric in the upper halfplane $\mathbb{U} = \{z = x + iy \in \mathbb{C} : y > 0\}$ is given infinitesimally at a point $z = x + iy \in \mathbb{U}$ by

$$ds^2_{\mathbb{U}} = \frac{dx^2 + dy^2}{y^2}, \qquad ds_{\mathbb{U}} = \frac{|dz|}{y}.$$

Given $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{U}$, the distance between them is

$$d_{\mathbb{U}}(z_1, z_2) = \log \frac{|z_1 - \overline{z_2}| + |z_1 - z_2|}{|z_1 - \overline{z_2}| - |z_1 - z_2|}.$$

The geodesic lines are precisely the intersections with $\mathbb{U}$ of circles orthogonal to the real line, plus rays perpendicular to the real line.

Both Poincaré models preserve hyperbolic angles, and are thereby conformal.

All isometries within these models are therefore Möbius transformations.

The halfplane model is “identical” (isometric) to the Poincaré disc model.

There is a simple and remarkable relationship between the angles and the area of a triangle which can be obtained as a consequence of Gauss-Bonnet formula:

The hyperbolic area of a triangle with interior angles α, β, γ is

$$\pi - (\alpha + \beta + \gamma) \leq \pi.$$

Then, Euclidean triangles are wider than hyperbolic triangles. One can think that the Euclidean plane is wider than the hyperbolic plane.

The area A_r of a hyperbolic disk of radius r is independent of the center, and is given by $4\pi \sinh^2(r/2)$.

The length L_r of the hyperbolic circle of radius r is $2\pi \sinh r$.

Therefore,

$$A_r \approx \pi r^2, \qquad L_r \approx 2\pi r, \qquad as \;\; r \to 0^+,$$

$$A_r \approx L_r \approx \pi e^r, \qquad as \;\; r \to \infty.$$

Hence, the hyperbolic plane is wider than the Euclidean plane (although Euclidean triangles are wider than hyperbolic ones).

In complex analysis, the most important property of the Poincaré metric is that holomorphic mappings are contractions with respect to it. More precisely, we have:

Theorem (Schwarz-Pick)

Every holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ verifies

$$d_{\mathbb{D}}\big(f(z_1), f(z_2)\big) \leq d_{\mathbb{D}}(z_1, z_2)$$

for every $z_1, z_2 \in \mathbb{D}$.
Furthermore, if the equality holds for some $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$, then f is an automorphism (i.e., a Möbius self-map of $\mathbb{D}$), and so $d_{\mathbb{D}}\big(f(z_1), f(z_2)\big) = d_{\mathbb{D}}(z_1, z_2)$ for every $z_1, z_2 \in \mathbb{D}$.

In fact, the Poincaré metric can be defined for any domain $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ has more than one point. If we denote by d_Ω the Poincaré distance in Ω, then we have the following generalization of Schwarz-Pick Lemma:

Theorem

If $\partial\Omega_1$ and $\partial\Omega_2$ have more than one point and $f : \Omega_1 \longrightarrow \Omega_2$ is holomorphic, then

$$d_{\Omega_2}\big(f(z_1), f(z_2)\big) \leq d_{\Omega_1}(z_1, z_2)$$

for every $z_1, z_2 \in \Omega_1$.
Furthermore, if the equality holds for some $z_1, z_2 \in \Omega_1$ with $z_1 \neq z_2$, then f is a conformal map of Ω_1 onto Ω_2, and so $d_{\Omega_2}\big(f(z_1), f(z_2)\big) = d_{\Omega_1}(z_1, z_2)$ for every $z_1, z_2 \in \Omega_1$.

The simpler particular case of Schwarz-Pick Lemma is the classical Schwarz's Lemma:

Theorem

Every holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ *with* $f(0) = 0$ *verifies*

$$\big|f(z)\big| \leq |z|$$

for every $z \in \mathbb{D}$.

This is a bound for the growth of every holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ (with the normalization $f(0) = 0$).

One of the most famous applications of the generalization of Schwarz-Pick Lemma is the following:

Theorem (Schottky's Theorem)

If $f : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0, 1\}$ *is holomorphic, then*

$$|f(z)| \leq \exp\left(\frac{1+|z|}{1-|z|}\left(7 + \max\left\{0, \log|f(0)|\right\}\right)\right),$$

for every $z \in \mathbb{D}$.

Note that this is a bound for the growth of every holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0, 1\}$.

HYPERBOLIC SPACES

Hyperbolic spaces

We say that $\gamma : [a, b] \longrightarrow X$ is a geodesic if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$.
A geodesic line is a geodesic whose domain is $\mathbb{R}$.

We say that X is a geodesic metric space if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient).

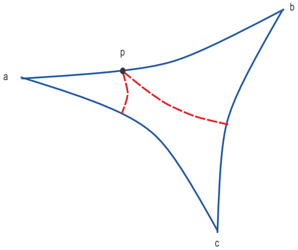
It is clear that every geodesic metric space is path-connected.

If the metric space X is a graph, we use the notation $[u, v]$ for the edge joining the vertices u and v.

Hyperbolic spaces

Definition

Let X be a geodesic metric space. A geodesic triangle $T = \{a, b, c\}$ is the union of three geodesics $[ab]$, $[bc]$ and $[ca]$ in X.



T is δ-thin if **for every** $p \in [ab]$ we have $d(p, [bc] \cup [ca]) \leq \delta$.

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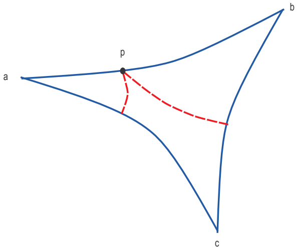
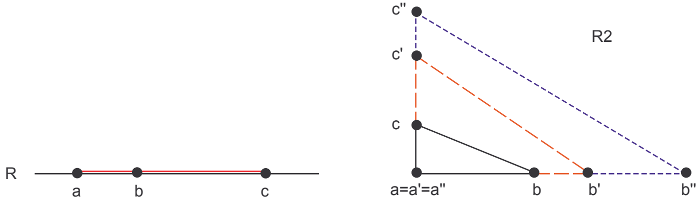


T is δ-thin if **for every** $p \in [ab]$ we have $d(p, [bc] \cup [ca]) \leq \delta$.

The space X is δ-hyperbolic if **every geodesic triangle** in X is δ-thin.

Examples

- $\mathbb{R}^n$ with the Euclidean metric is hyperbolic if and only if $n = 1$.



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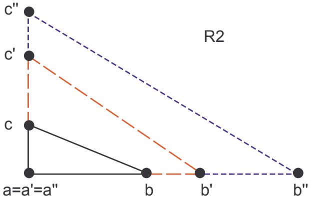
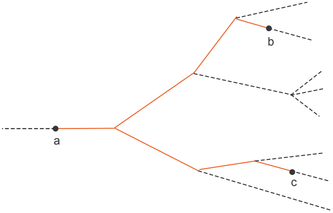


- Every tree is 0-hyperbolic.



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- Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic.
- The unit disk $\mathbb{D}$, with its Poincaré metric, is $\log\left(1+\sqrt{2}\right)$-hyperbolic.
- Every simply connected complete Riemannian manifold with sectional curvature satisfying $K \leq -k$, with $k > 0$, is hyperbolic.

Examples III

- The graph Γ of the routing infrastructure of the Internet is also empirically shown to be hyperbolic.

One can think that this is a trivial (and then a non-useful) fact, since every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic.

The point is that the quotient

$$\frac{\delta(\Gamma)}{\operatorname{diam} \Gamma}$$

is very small, and this makes the tools of hyperbolic spaces applicable to Γ.

When is a space hyperbolic?

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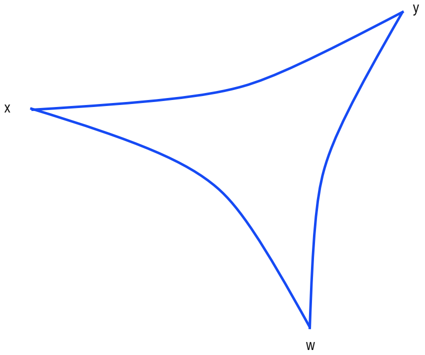
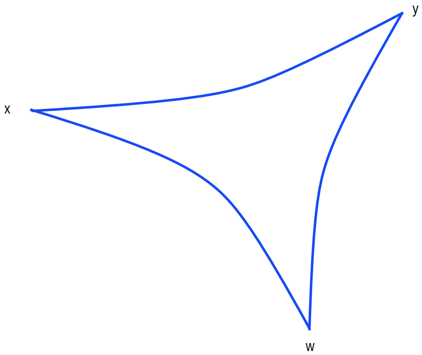


When is a space hyperbolic?

Step 2. *For each* $P \in \mathbf{T}$, *compute* $\mathrm{dist}(P, \mathbf{A})$ *being* $\mathbf{A}$ *the union of the other sides of* $\mathbf{T}$.



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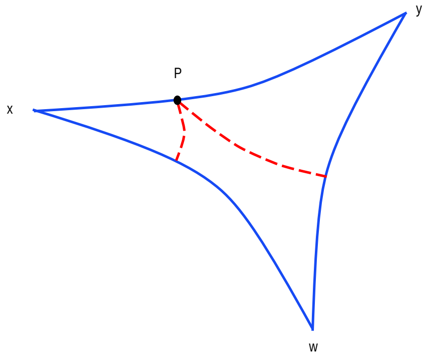

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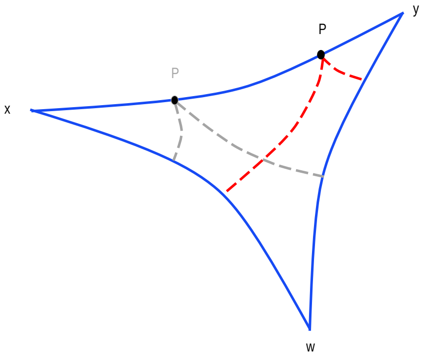

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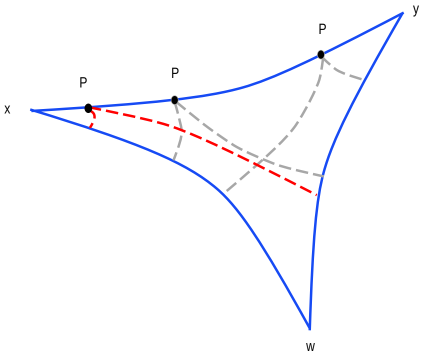

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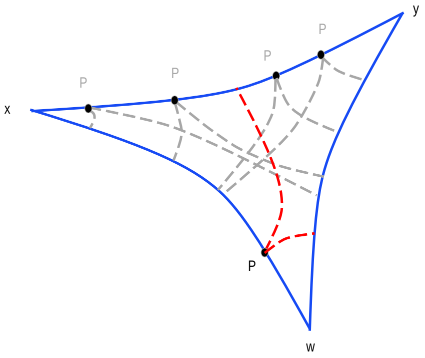
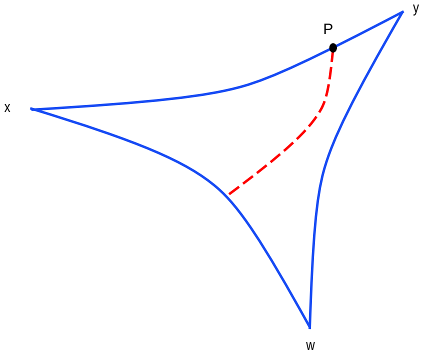


When is a space hyperbolic?

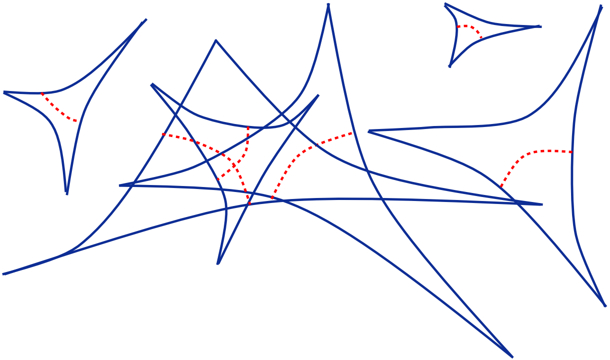
Step 3. *Compute* $\delta(\mathbf{T}) := \max_{P}(\text{dist}(P, A))$.



When is a space hyperbolic?

We need to repeat Steps 1, 2 and 3 for every geodesic triangle in the space!

$$\delta(\mathbf{X}) := \sup_{T} \delta(T)$$



Metric graphs

By a graph G we mean a set of points called vertices connected by (undirected) edges; the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$; we assume also that each edge has a length assigned to it.

In order to consider a graph G as a geodesic metric space, we identify (by an isometry) any edge $[u, v] \in E(G)$ with the real interval $[0, \ell]$ (if $\ell := L([u, v])$); therefore, any point in the interior of an edge is a point of G.

Then G is naturally equipped with a distance defined on its points, induced by taking the shortest paths in G, and we see G as a metric graph.

Metric graphs

We allow loops and multiple edges in the graphs; we also allow edges of arbitrary lengths.

We always consider graphs which are connected and locally finite (i.e., in each ball there are just a finite number of edges).

These properties guarantee that the graphs are geodesic metric spaces.

In what follows, although the results will be stated for hyperbolic spaces, you can think that they are always hyperbolic graphs, by simplicity.

TWO MAIN RESULTS

Result 1: invariance of hyperbolicity.

If X and Y are metric spaces, a function $f : X \longrightarrow Y$ is an *(a, b)-quasi-isometric embedding*, with $a \geq 1$ and $b \geq 0$, if

$$\frac{1}{a}\, d_X(x, y) - b \leq d_Y(f(x), f(y)) \leq a\, d_X(x, y) + b\,, \qquad \text{for every } x, y \in X\,.$$

The function f is *ε-full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \geq 1,\ \beta, \varepsilon \geq 0$ such that f is an ε-full (α, β)-quasi-isometric embedding.

Result 1: invariance of hyperbolicity.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \longrightarrow Y$.

Being quasi-isometric is an equivalence relation.

An (α, β)-*quasigeodesic* is an (α, β)-quasi-isometric embedding whose domain is an interval on the real line.

Result 1: invariance of hyperbolicity.

In the study of any mathematical property, the class of maps which preserve that property plays a central role in the theory. The following result shows that quasi-isometries preserve hyperbolicity.

Theorem

Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y.

If Y is hyperbolic, then X is hyperbolic.

Besides, if f is ε-full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

Result 2: geodesic stability.

Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call *ε-neighborhood* of Y in X, denoted by $V_\varepsilon(Y)$ to the set $\{x \in X : d_X(x, Y) \leq \varepsilon\}$.

The *Hausdorff distance* between two subsets Y and Z of X, denoted by $\mathcal{H}(Y, Z)$, is the number defined by:

$$\inf\{\varepsilon > 0 : Y \subset V_\varepsilon(Z) \textit{ and } Z \subset V_\varepsilon(Y)\}.$$

Result 2: geodesic stability.

In the complex plane (with its Euclidean distance), there is only one optimal way of joining two points: a straight line segment.

However if we allow "limited suboptimality", the set of "reasonably efficient paths" (quasigeodesics) are well spread.

For instance, if we split the circle $\partial D(0, R) \subset \mathbb{C}$ into its two semicircles between the points R and $-R$, then we have two reasonably efficient paths (two $(\pi/2, 0)$-quasigeodesics) between these endpoints such that the point Ri on one of the semicircles is far from all points on the other semicircle provided that R is large.

Result 2: geodesic stability.

Even an additive suboptimality can lead to paths that fail to stay close together. For instance, the union of the two line segments in $\mathbb{C}$ given by $[0, R + i\sqrt{R}\,]$ and $[R + i\sqrt{R}\,, 2R]$ gives a path of length less than $2R + 1$ (since $2\sqrt{R^2 + R} \leq 2R + 1$), and so is "additively inefficient" by less than 1 (a $(1,1)$-quasigeodesic).

However, its corner point is very far from all points on the line segment $[0, 2R]$ when R is very large.

The situation in Gromov hyperbolic spaces is very different, since all such reasonably efficient paths ((α, β)-quasigeodesics for fixed α, β) stay within a bounded distance of each other:

Result 2: geodesic stability.

Theorem

For every $\delta \geq 0$, $a \geq 1$ and $b \geq 0$, there exists a constant H_0, which just depends on δ, a and b, verifying the following property: Let X be a δ-hyperbolic geodesic metric space and let g, γ (a, b)-quasigeodesics with the same endpoints. Then $\mathcal{H}(g, \gamma) \leq H_0$.

In fact, the geodesic stability is equivalent to the hyperbolicity (*M. Bonk*).

GROMOV PRODUCT

Gromov product

If X is a metric space, we define the *Gromov product* of $x, y \in X$ with base point $w \in X$ by

$$(x, y)_w := \frac{1}{2}\left(d(x, w) + d(y, w) - d(x, y)\right).$$

A geometric interpretation of the Gromov product is obtained by mapping the triple (x, y, w) isometrically onto a triple (x', y', w') in the Euclidean plane.

The circle inscribed to the triangle x', y', w' meets the sides $[w'x']$ and $[w'y']$ at points x^* and y^*, respectively, and we have $(x, y)_w = |x^* - w'| = |y^* - w'|$.

Gromov product

If X is a Gromov hyperbolic space, it holds

$$(x, z)_w \geq \min \left\{(x, y)_w, (y, z)_w\right\} - \delta^* \qquad (1)$$

for every $x, y, z, w \in X$ and some constant $\delta^* \geq 0$. In fact, we have:

Theorem

Let us consider any geodesic metric space X.
(i) If X is δ-hyperbolic, then it satisfies (1) with constant $\delta^* = 3\delta$.
(ii) If X satisfies (1) with constant δ^*, then it is $3\delta^*$-hyperbolic.

Hence, if X is a geodesic metric space, then (1) is equivalent to our definition of Gromov hyperbolicity.

Gromov product

Then (1) extends the definition of Gromov hyperbolicity to the context of (non-necessarily geodesic) metric spaces.

The disadvantage of the Gromov product definition is that its geometric meaning is unclear at first sight, whereas the thin triangles definition is very easy to understand geometrically.

Gromov product

The following useful estimate is the key to understand the geometric meaning of the Gromov product definition (1):

Proposition

If X is a δ-hyperbolic geodesic metric space, then for every $x, y, w \in X$ and every geodesic $[xy]$, we have

$$d(w, [xy]) - 4\delta \leq (x, y)_w \leq d(w, [xy]).$$

Indeed only the lower bound requires hyperbolicity.

THE IMPORTANCE OF HYPERBOLIC SPACES

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- Secure transmission of information on the internet.
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- Traffic flow and congestion.
- Large-scale data visualization.

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- Hyperbolic groups are strongly geodesically automatic (there is an automatic structure on the group, where the language accepted by the word acceptor is the set of all geodesic words).
- The hyperbolicity is also useful in the study of DNA data.

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- They are useful in Group Theory.
- It is possible to consider angles in metric spaces!
- The hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is.
- Geodesic stability: any geodesic and quasigeodesic with the same endpoints are (uniformly) close.
  The geodesic stability is equivalent to the hyperbolicity (*M. Bonk*).

Pythagorean theorem for the hyperbolic plane (relation among the three sides of a right-angled triangle):

$\cosh c = \cosh a \cosh b$

Cosine rule for the hyperbolic plane:

$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\theta$

If $a, b, c >> 1$, then

$\frac{1}{2}e^c \approx \frac{1}{4}e^{a+b}(1 - \cos\theta)$

$e^c \approx e^{a+b}\sin^2(\theta/2) \Longrightarrow \frac{1}{2}(a + b - c) \approx \log\frac{1}{\sin(\theta/2)}$

$(x, y)_w \approx \log\frac{1}{\sin(\theta/2)}$

In the setting of linear spaces with inner product, the main angle is $\pi/2$; however, in hyperbolic spaces the main angle is π: One can check that, in a geodesic metric space, $(x, y)_w = 0$ if and only if w belongs to a geodesic joining x and y.
The following result is a good example of "angle estimation" in hyperbolic spaces:

Theorem

Let us consider constants $\delta \geq 0, r, \ell > 0$, a δ-hyperbolic geodesic metric space X and a finite sequence $\{x_j\}_{0 \leq j \leq n}$ in X with

$$d(x_{j-1}, x_{j+1}) \geq \max\{d(x_{j-1}, x_j), d(x_j, x_{j+1})\} + 18\delta + r, \quad 0 < j < n,$$
$$d(x_{j-1}, x_j) \leq \ell, \qquad 0 < j \leq n.$$

Then $[x_0 x_1] \cup [x_1 x_2] \cup \cdots \cup [x_{n-1} x_n]$ is an $(\alpha, 0)$-quasigeodesic, with $\alpha := \max\{\ell, 1/r\}$.

Why are hyperbolic spaces important (from a mathematical point of view)?

- Isometries (and quasi-isometries) can be extended (as an homeomorphism) to the Gromov boundary of the space. This fact allows to classify the isometries as *elliptic*, *hyperbolic* and *parabolic*, as the Möbius maps in $\mathbb{D}$.
  A main ingredient in the proof of this result in the unit disk $\mathbb{D}$ is that the isometries are holomorphic functions. Surprisingly, the tools in hyperbolic spaces provide a new and general proof just in terms of distances!

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- This fact let us generalize Fefferman's Theorem about continuous extension up to the boundary of biholomorphic maps [*Invent. Math.*, 1974].

Are hyperbolic graphs important (from a mathematical point of view)?

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- Besides, many negatively curved surfaces are quasi-isometric to a very simple graph [Tourís, *J. Math. Anal. Appl.*, 2011].

BETTER BOUNDS

Theorem

Let X be a δ-hyperbolic geodesic metric space, $u, v \in X$, b a non-negative constant, h a curve joining u and v with $L(h) \leq d(u, v) + b$, and g any geodesic joining u and v. Then,

$$h \subseteq V_{8\delta+b/2}(g), \quad g \subseteq V_{16\delta+b}(h), \quad \mathcal{H}(g, h) \leq 16\delta + b.$$